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FAST TRACK COMMUNICATION

Zero width resonance (spectral singularity) in a complex *PT*-symmetric potential

Zafar Ahmed

Nuclear Physics Division, Bhabha Atomic Research Centre, Bombay 400 085, India

E-mail: zahmed@barc.gov.in

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Abstract

We show that the complex *PT*-symmetric potential $V(x) = -V_1\,\mathrm{sech}^2 x + \mathrm{i}V_2\,\mathrm{sech}\,x\tanh x$ entails a single zero-width resonance (spectral singularity) when $V_1 + |V_2| = 4n^2 + 4n + \frac{3}{4}(n = 0, 1, 2, 3, \ldots, |V_2| > V_1 + 1/4)$ and the positive resonant energy is given as $E_* = \frac{1}{4}[|V_2| - (1/4 + V_1)]$.

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The physical energy poles of *s*-matrix or transmission/reflection amplitudes yield a discrete spectrum of bound states and resonances of a Hermitian scattering potential well [1]. In the former, the energies are real and negative (within the potential well), whereas in the latter, these are complex where the real part is positive. An interesting study of the physical poles of scattering amplitudes for a versatile and exactly solvable potential is available in [2]. Resonances are also called Gamow or Sigert states embedded in positive energy continuum which were first applied in the theory of alpha-decay [1].

For about a decade [3, 4] now, the non-Hermitian complex *PT*-symmetric potentials have been investigated to have a real discrete spectrum. In the *PT*-symmetry *P* denotes parity transformation $(x \to -x)$ and *T* the time-reversal: $(\mathrm{i} \to -\mathrm{i})$.

The complex *PT*-symmetric potential

$$V(x) = -V_1\,\mathrm{sech}^2 x + \mathrm{i}V_2\,\mathrm{sech}x\tanh x \qquad (1)$$

is the first [5] exactly solvable model of the complex *PT*-symmetric potential to demonstrate analytically and explicitly that the spectrum is real and discrete provided $V_2 < V_1 + 1/4$ (assuming V_1 to be positive) and the energy eigenstates are also the eigenstates of the antilinear operator *PT*; otherwise the *PT*-symmetry is spontaneously broken and the spectrum contains complex conjugate pairs of eigenvalues. This model has helped in finding or demonstrating several other features of complex *PT*-symmetric interactions [6, 7].

The real Hermitian version of this scattering potential is called Scarf II for which the exact analytic scattering amplitudes have already been found [8, 9]. In this communication, we would like to show that the reflection/transmission amplitudes for (1) when $V_1 > 0$ have two kinds of discrete poles. One set of them are having real and complex-conjugate energies with real part as negative. These are otherwise known as a discrete spectrum of bound states [5].

The other one is a single positive energy which exists provided the potential parameters V_1, V_2 satisfy a certain special condition. This is quite like the shape resonance embedded in positive energy continuum of a Hermitian potential. However, in contrast, the new resonance is having a zero width. In recent instructive investigations [10, 11], this pole has been discussed as a spectral singularity of non-Hermitian Hamiltonian which is also like a resonance with zero width. In [11], as an example, a complex *PT*-symmetric model has been used to find the spectral singularity; the calculations are very cumbersome and implicit. In the following, we present the potential (1) as an exactly solvable model for the spectral singularity. Here both the condition on the potential parameters and the resonant energy are very simple and explicit.

Using $2m = 1 = \hbar^2$ for the Schrödinger equation, let us define

$$k = \sqrt{E}, \quad p = \tfrac{1}{2}\sqrt{|V_2| + V_1 + 1/4}, \quad q = \tfrac{1}{2}\sqrt{|V_2| - (V_1 + 1/4)}, \quad s = \tfrac{1}{2}\sqrt{1/4 + V_1 - |V_2|}, \tag{2}$$

where E is the energy. Then following [8, 9], we can write the transmission amplitude for (1) as

$$t(k) = \frac{\Gamma[1/2 - p - \mathrm{i}(q+k)]\Gamma[1/2 + p + \mathrm{i}(q-k)]\Gamma[1/2 + p - \mathrm{i}(q+k)]\Gamma[1/2 - p + \mathrm{i}(q-k)]}{\Gamma[-\mathrm{i}k]\Gamma[1+\mathrm{i}k]\Gamma^2[1/2 - \mathrm{i}k]},$$
$$r(k) = f(k)t(k),$$
$$f(k) = \mathrm{i}[(\sin^2 \pi p + \sinh^2 \pi q)\,\mathrm{sech}\,\pi k + (\cos^2 \pi p + \sinh^2 \pi q)\,\mathrm{cosech}\,\pi k]. \tag{3}$$

Earlier for non-symmetric complex potentials handedness of reflectivity has been proved [12]. For the complex *PT*-symmetric potential (1), $r(-k) \neq r(k)$ follows consequently [6, 12].

The property of the Gamma (Γ) function that $\Gamma(-N) = \infty$ where N is a non-negative integer helps in studying the poles of the transmission amplitude $t(k)$ (3). The poles of four Gamma functions in (3) are $\mathrm{i}k_n = [n + 1/2 + \epsilon p + \sigma \mathrm{i}q]$, where $n = 0, 1, 2, \ldots,$ and ϵ and σ are $\pm$ independently. Consequently, we recover a discrete spectrum of complex conjugate pairs [5] from the relevant ($\epsilon = -$) poles when $|V_2| > 1/4 + V_1$, $V_1 > 0$:

$$E_n = -[n + 1/2 - (p \pm \mathrm{i}q)]^2, \tag{4}$$

as s is real. When $|V_2| < V_1 + 1/4$, $V_1 > 0$ and s is purely imaginary, we recover [5] two branches of the real discrete spectrum:

$$E_{n^+} = -[n^+ + 1/2 - (p+s)]^2, \quad E_{n^-} = -[n + 1/2 - (p-s)]^2, \quad n^\pm = 0, 1, 2, 3, \ldots, m^\pm. \tag{5}$$

Here $m^\pm$ = Integer part of $[p \pm s]$. Note that in these eigenvalues ((4), (5)), the real part is negative. When $V_1 < 0$, s ((2), (5)) becomes non-real and there are no real bound states. More importantly in this case, the real part of (1) becomes a barrier [6]. However, in what follows V_1 could be positive or negative.

Now we find a very interesting scope for the poles of (3) at positive discrete energies. We set

$$1/2 - p = -n \qquad \text{or} \qquad 1/2 + p = -n, \quad n \geqslant 0. \tag{6}$$

We get a condition on the potential parameter as

$$V_1 + |V_2| = 4n^2 + 4n + \tfrac{3}{4}, \qquad n = 0, 1, 2, 3, \ldots. \tag{7}$$

Further, we get $k_* = \pm q$ or equivalently

$$E_* = \tfrac{1}{4}[|V_2| - (1/4 + V_1)] \geqslant 0, \tag{8}$$

which is a single energy. The presence of $|\cdot|$ indicates the commonness of these results ((7), (8)), even if the sign of V_2 is changed. Changing the sign of V_2 in (1) is equivalent to changing the direction of incidence of the particle at the potential. In doing so, as said earlier, only the reflection amplitudes will change and not the poles. Also note that for the positivity of E_* and hence for the existence of the spectral singularity, $|V_2|$ needs to be larger than $V_1 + 1/4$, meaning the imaginary part of the potential (1) should be stronger than the real part. However, for the binding potentials such as $V(x) = x^2 - g(\mathrm{i}x)^\nu$ whose both real and imaginary parts diverge asymptotically, the concept of the stronger real/imaginary part may not make sense.

So we conclude that whenever the complex *PT*-symmetric scattering potential (1) has its parameters satisfying condition (7), there will occur a zero-width single resonance at real energy $E = E_*$ (8). We conjecture that for complex *PT*-symmetric scattering potentials (s.t., $V(\pm\infty) = 0$) the imaginary part of the potential ought to be necessarily stronger than that of the real part. It is instructive to note here that whether or not the real part of the complex *PT*-symmetric potential is a well or a barrier the parameter-dependent spectral singularity can occur. Curiously enough like the model of [11], here too we get a single resonant energy (8) when the potential (1) is fixed as per condition (7). Therefore, further, it is desirable to investigate whether a complex *PT*-symmetric potential can support at most one (or more) spectral singularity(ies).

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